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RESIDUAL BEHAVIOR OF INDUCED MAPS

BY

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ABSTRACT

Consider $(X, \mathcal{F}, \mu, T)$ a Lebesgue probability space and measure preserving invertible map. We call this a dynamical system. For a subset $A \in \mathcal{F}$, by T_A: $A \to A$ we mean the induced map, $T_A(x) = T^{r_A(x)}(x)$ where $r_A(x) = \min\{i > 0: T^i(x) \in A\}$. Such induced maps can be topologized by the natural metric $D(A, A') = \mu(A \triangle A')$ on $\mathcal{F}$ mod sets of measure zero. We discuss here ergodic properties of T_A which are residual in this metric. The first theorem is due to Conze.

THEOREM 1 (Conze): *For T ergodic, T_A is weakly mixing for a residual set of A.*

THEOREM 2: *For T ergodic, 0-entropy and loosely Bernoulli, T_A is rank-1 and rigid for a residual set of A.*

THEOREM 3: *For T ergodic, positive entropy and loosely Bernoulli, T_A is Bernoulli for a residual set of A.*

THEOREM 4: *For T ergodic of positive entropy, T_A is a K-automorphism for a residual set of A.*

A strengthening of Theorem 1 asserts that A can be chosen to lie inside a given factor algebra of T. We also discuss even Kakutani equivalence analogues of Theorems 1–4.

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Introduction

Much work has been done to understand the residual dynamics of invariant measures for homeomorphisms of a compact metric space. In particular suppose Σ is an infinite cartesian product $\Sigma = \bigotimes_{i=-\infty}^{\infty} \Sigma_i$ where each Σ_i is a copy of some fixed compact metric space (Σ_0, m), and let σ be the left shift on this space Σ of sequences. Let $\mathcal{M}(\Sigma,\sigma)$ be the space of all σ-invariant borel probability measures. $\mathcal{M}(\Sigma,\sigma)$ is a compact, convex metric space in the weak* topology and its extreme points are the ergodic measures. Let $\hat{\mathcal{M}}(\Sigma,\sigma) = \{(\Sigma,\mathcal{B},\nu,\sigma)\colon \nu \in \mathcal{M}(\Sigma,\sigma)\}$ topologized with the weak* topology inherited from $\mathcal{M}(\Sigma,\sigma)$.

The following result is essentially well-known, and follows easily from the corresponding statement about the usual weak topology on the group of all measure-preserving map (see e.g. [H][K,S]).

Proposition 1: *A residual set of systems $(\Sigma,\mathcal{B},\nu,\sigma) \in \hat{\mathcal{M}}(\Sigma,\sigma)$ are weakly mixing, rank-1 and rigid.*

We wish to consider a different topological setting in which to examine residual behavior, the set of induced maps for a particular dynamical system $(X,\mathcal{F},\mu,T)$. Consider the collection of such induced maps T_A acting on A with renormalized measure $\mu|_A$. These form a class of dynamical systems metrized by the natural metric $D(A,A') = \mu(A \triangle A')$. Call this metric space of dynamical systems $\mathcal{M}_T$.

Residuality in this space was first studied by Conze [C], who proved that for T ergodic the weakly mixing systems are residual in $\mathcal{M}_T$ (Theorem 1 above). We will prove (Theorems 2, 3, 4) that in the appropriate situations the rank one and rigid systems, the Bernoulli systems and the K systems, respectively, are residual in $\mathcal{M}_T$. We also include a proof of Theorem 1, as it follows immediately from the density of the weakly mixing systems [FO] (essentially this was known earlier — see [Ch]) together with a simple general principle, Lemma 3 below, which is also used in the proof of Theorem 2.

In Theorems 2, 3 and 4 the density of the relevant class was already known ([O,R,W],[O,S]); what is new here is that these classes are G_δ's. Theorem 1 has the following useful strengthening.

Theorem 5: *Suppose T ergodic and $\mathcal{G} \subset \mathcal{F}$ is a non-atomic factor algebra for T. Then the class of A's in $\mathcal{G}$ for which T_A is weakly mixing, as a transformation on $\mathcal{F}$, is residual in $\mathcal{G}$.*

In fact a simple observation about the proof in [C] of Theorem 1 actually gives a proof of Theorem 5 as well — this was pointed out to us by J.-P. Thouvenot. We give a quite different and completely self-contained proof of Theorem 5 (and hence also of Theorem 1). As an example of an application of Theorem 5 we mention the following result which follows from Theorems 3 and 5, [O,R,W] and [R].

COROLLARY 6: *Suppose that $(X, \mathcal{F}, \mu, T)$ is a loosely Bernoulli positive entropy system, $\mathcal{G} \subset \mathcal{F}$ is a factor algebra for T and T is relatively isometric as an extension of its restriction to $\mathcal{G}$. Then the A's in $\mathcal{G}$ such that T_A is Bernoulli on $\mathcal{F}$ form a residual class in $\mathcal{G}$*

Theorems 1–5 all have analogues in the context of even Kakutani equivalence. Recall ([F,J,R], [J,R]) that a time change S of T is a map having a.e. the same orbits as T, and S is said to be a Kakutani time change of T if there is a non-null $A \in \mathcal{F}$ such that $T_A = S_A$. We denote by $K(T)$ the space of Kakutani time changes of T. $K(T)$ carries a complete metric d defined by

$$d(S, S') = 1 - \sup\{\mu(A): S_A = S'_A\}.$$

THEOREM 1′: *For T ergodic $\{S \in K(T)$: S is weakly mixing $\}$ is residual in $K(T)$.*

THEOREM 2′: *For T loosely Bernoulli of zero entropy $\{S \in K(T)$: S is rank 1 and rigid$\}$ is residual in $K(T)$.*

THEOREM 3′: *For T loosely Bernoulli of positive entropy $\{S \in K(T)$: S is Bernoulli$\}$ is residual in $K(T)$.*

THEOREM 4′: *For T ergodic of positive entropy $\{S \in K(T)$: S is a K-automorphism$\}$ is residual in $K(T)$.*

If $\mathcal{G}$ is a factor algebra for T, a time change S of T is called $\mathcal{G}$-measurable if $S\mathcal{G} = \mathcal{G}$, that is $Sx = T^{n(x)}x$ with $n(\cdot)$ $\mathcal{G}$-measurable. We denote by $K_{\mathcal{G}}(T)$ the set of $\mathcal{G}$-measurable time changes S of T such that $S_A = T_A$ for some A in G. $K_{\mathcal{G}}(T)$ is a closed subset of $K(T)$.

THEOREM 5′: *$\{S \in K_{\mathcal{G}}(T)$: S is weakly mixing$\}$ is residual in $K_{\mathcal{G}}(T)$.*

Theorems 3′ and 5′ are used in [F,J,R] in the proof of a Kakutani equivalence version of the relative isomorphism theorem of [R].

We will prove only the unprimed theorems. The proofs of the primed theorems are exactly parallel, although there is no apparent method of deducing one set of theorems from the other.

Our proof of Theorem 3 is short, but it invokes the Kakutani equivalence theory [O,R,W] for the required density. However, Fieldsteel [F] has independently found a self-contained proof which in fact reduces the proof of the equivalence theorem, at least in the positive entropy case, to the isomorphism theory for Bernoulli shifts.

Our first step will be to show that M_T can be continuously embedded in a shift space $\hat{M}(\Sigma,\sigma)$ for an appropriate choice of Σ_0. It will follow that any subset of $\hat{M}(\Sigma,\sigma)$ which is a G_δ is also a G_δ when viewed, via pullback, as a subset of M_T. Theorems 1 and 2 then follow almost for free from Proposition 1.

Embedding $\mathcal{M}_T$ in a shift space

Fix a dynamical system $(X,\mathcal{F},\mu,T)$ and let $\{P_i\}_{i=1}^{\infty}$ be a refining and generating sequence of partitions. Let $P_i = \{P_{(i,1)}, P_{(i,2)}, \dots, P_{(i,k_i)}\}$ and for each point $x \in X$, $x \in C_i(x) = P_{(i,j(x,i))}$ where $C_{i+1}(x) \subseteq C_i(x)$ and $\bigcap_{i=1}^{\infty} C_i(x) = \{x\}$.

Set

$$\Sigma_0 = \bigotimes_{i=1}^{\infty}\{1,2,\dots,k_i\}$$

with the compact metric product topology. A point $s \in \Sigma_0$ is a function $s\colon \mathbb{N} \to \mathbb{N}$ such that $s(i) \in \{1,\dots,k_i\}$. We can define an injection $\phi\colon X \to \Sigma_0$ by

$$\phi(x)(i) \;=\; j(x,i).$$

Thus $\phi(x)$ is simply the sequence of subscripts of the branch in the tree of partitions $\{P_i\}$ intersecting to x. ϕ is a borel map in that $\phi^{-1}(\mathcal{B}) \subseteq \mathcal{F}$. Moreover $\phi^{-1}(\mathcal{B}) = \mathcal{F}$ modulo sets of μ-measure zero. Set $\Sigma = \bigotimes_{i=-\infty}^{\infty} \Sigma_i$, where each Σ_i is just a copy of Σ_0, and let σ be the left shift on the sequences of Σ.

For any set $A \in \mathcal{F}$ let $\phi_A\colon A \to \Sigma$ be given by $(\phi_A(x))_i = \phi(T_A^i(x))$. Notice $\phi_A T_A(x) = \sigma\phi_A(x)$, i.e. ϕ conjugates (T_A, X) to $(\sigma, \phi(X))$. Let $\nu_A \in \mathcal{M}(\Sigma,\sigma)$ be $\phi_A^*(\mu)$, an ergodic σ-invariant measure.

Lemma 1: *The dynamical system $(\Sigma,\mathcal{B},\nu_A,\sigma)$ is measurably conjugate to $(A,\mathcal{F}|_A,\mu|_A,T_A)$ by the map ϕ_A.*

COROLLARY 2: *The map ψ: $\mathcal{F} \to \mathcal{M}(\Sigma,\sigma)$ given by $\psi(A) = \nu_A$ is one to one mod D, i.e. if $\psi(A) = \psi(A')$ then $D(A,A') = 0$.*

Proof: If $\nu_A = \nu_{A'}$, then for all sets $\bar{A}$ which are P_i-measurable for some i,

$$\frac{\mu(A\cap\bar{A})}{\mu(A)} = \frac{\mu(A'\cap\bar{A})}{\mu(A')}.$$

We can find sets $\bar{A}_i$ with

$$\frac{\mu(A\cap\bar{A}_i)}{\mu(A)} \xrightarrow{i} 1 \quad \text{and} \quad \frac{\mu(A\cap\bar{A}_i^c)}{\mu(A)} \xrightarrow{i} 0,$$

i.e. $\mu(A\triangle\bar{A}_i) \xrightarrow{i} 0$. But then $\mu(A'\triangle\bar{A}_i) \xrightarrow{i} 0$ and $D(A,A') = 0$.

The map ψ: $\mathcal{F} \to \mathcal{M}(\Sigma,\sigma)$ lifts to a map $\hat{\psi}$: $\mathcal{M}_T \to \hat{\mathcal{M}}(\Sigma,\sigma)$ where $\hat{\psi}(A, \mathcal{F}|_A, \mu|_A, T_A) = (\Sigma, \mathcal{B}, \psi(A), \sigma)$.

LEMMA 3: *$\hat{\psi}$ is continuous, hence the inverse images of open sets in $\hat{\mathcal{M}}(\Sigma,\sigma)$ are open sets in $\mathcal{M}_T$.*

Proof: Denote by $Q_{(i,j)} \subset \Sigma_0$ the set $\{s \in \Sigma_0\colon\ s(i) = j\}$. The topology on $\mathcal{M}(\Sigma,\sigma)$ is generated by the functions

$$f_{i,j,k}(\nu) = \nu(\{\sigma \in \Sigma\colon s(k) \in Q_{(i,j)}\}).$$

Now

$$f_{(i,j,k)} \circ \psi(A) = \mu(T_A^k(P_{(i,j)}))$$

which is continuous in $\mathcal{M}_T$ as a function of A.

COROLLARY 4: *Any G_δ-subset in $\hat{\mathcal{M}}(\Sigma,\sigma)$ pulls back via $\hat{\psi}^{-1}$ to a G_δ-subset of $\mathcal{M}_T$.*

Proof of Theorem 1: Let $W \subseteq \hat{\mathcal{M}}(\Sigma,\sigma)$ be the weakly mixing systems. This is a dense G_δ. $\hat{\psi}^{-1}(W)$ consists of the weakly mixing elements of $\mathcal{M}_T$. Friedman and Ornstein [F,O] have shown that even the mixing elements of $\mathcal{M}_T$ are dense and so $\hat{\psi}^{-1}(W)$ is residual. (Density of the weakly mixing elements is actually much easier — it essentially follows from the construction in [Ch], as well as our proof of Theorem 5 below.)

Proof of Theorem 2: Let $R \subseteq \hat{\mathcal{M}}(\Sigma,\sigma)$ consist of the rank-1 rigid systems. This is a residual set [K,S]. Certainly $\hat{\psi}^{-1}(R)$ consists of the rank-1 rigid elements of $\mathcal{M}_T$. This set is empty unless T is 0-entropy and loosely Bernoulli. But if it is 0-entropy and loosely Bernoulli, $\hat{\psi}^{-1}(R)$ is dense [O,R,W].

THEOREMS 3 AND 4. In both proofs we will use the fact that $h(T_A, P|_A)$ is a continuous function of A. The proof is a standard name counting argument very similar to [O,R,W] Proposition 3.4.

Proof of Theorem 3: Let us say a process (S,P) is ε-F.D. if there is a $\delta > 0$ such that, whenever (S', P') is a process whose entropy and distribution are within δ of those of (S,P), then (S',P') is within ε of (S,P) in the $\bar{d}$-metric. Of course, finitely determined is simply ε-F.D. for every ε. Now given a finite partition P of X and $\varepsilon > 0$ let

$$\mathcal{O}(P,\varepsilon) \;=\; \{A \in \mathcal{F}\colon (T_A, P|_A) \text{ is } \varepsilon\text{-}F.D.\}.$$

We claim that $\mathcal{O}(P) = \bigcap_{\varepsilon>0} \mathcal{O}(P,\varepsilon)$ is a G_δ. To see this it suffices to show that each $A \in \mathcal{O}(P,\varepsilon)$ has an open neighborhood contained in $\mathcal{O}(P, 2\varepsilon)$. We know $\exists\delta > 0$ such that if (S',P') is within δ of $(T_A, P|_A)$ in entropy and distribution then it is within ε in $\bar{d}$. Since the distribution and entropy of $(T_A, P|_A)$ are both continuous functions of A, if A' is sufficiently close to A then $(T_A, P|_{A'})$ will be within $\delta/2$ of $(T_A, P|_A)$ in entropy and distribution, hence within ε in $\bar{d}$. This means that if (S', P') is within $\delta/2$ of $(T_{A'}, P|_{A'})$ in entropy and distribution then it is within ε of $(T_A, P|_A)$ in $\bar{d}$, hence within 2ε of $(T_{A'}, P_{A'})$ in $\bar{d}$, establishing our claim.

Finally, taking the intersection of $\mathcal{O}(P)$ over a countable dense collection of P's we get precisely those A's for which T_A is Bernoulli, since the finitely determined processes are closed in $\bar{d}$. Thus the Bernoulli's in $\mathcal{M}_T$ are a G_δ, and they are dense by the Kakutani equivalence theorem [O,R,W] (this is the only place where we use that T is loosely Bernoulli).

Our proof of Theorem 4 has a vaguely similar flavor, resting as it does on certain residual properties of $\hat{\mathcal{M}}(\Sigma,\sigma)$ and continuity of entropy within $\mathcal{M}_T$. Specifically, for any partition P, $h(T_A, P|_A)$ is a continuous function of A, but in $\hat{\mathcal{M}}(\Sigma,\sigma)$, $h_\mu(\sigma, P)$ is only upper semi-continuous, i.e. if $\mu_i \xrightarrow{i} \mu$ then

$$\overline{\lim}\, h_{\mu_i}(\sigma, P) \le h_\mu(\sigma).$$

Theorem 4 will follow with relative ease from the following result.

PROPOSITION 2: *For any $k \in N$, and finite partition P, $h((T_A)^k, P|_A)$ is a continuous function of A.*

We begin the proof of this proposition with a small lemma analogous to upper semi-continuity of $h_\mu(\sigma, P)$ in μ.

LEMMA 6: *For any subset $I \subseteq \mathbb{Z}$,*

$$f(\mu) = h_\mu(P_1 | \bigvee_{i\in I} \sigma^{-i}(P_1))$$

is upper semi-continuous in $\mathcal{M}(\Sigma, \sigma)$.

Proof: Clearly

$$f_N(\mu) = h_\mu(P_1 | \bigvee_{i\in I\cap[-N,N]} \sigma^{-i}(P_1))$$

is continuous in μ, and further,

$$f_{N+1}(\mu) \le f_N(\mu) \quad \text{and} \quad f_N(\mu) \xrightarrow{N} f(\mu).$$

The result follows.

Fix $k \in \mathbb{Z}$ and define in $\mathbb{Z}$ sets $I_0, I_1, \ldots, I_{k-1}$ by

$$I_0 = \{-ik\colon i > 0\} \quad \text{and} \quad I_j = I_0 \cup \{-t - ik\colon t \in \{1, 2, \ldots, j\} \text{ and } i \in \mathbb{Z}\}.$$

What I_j consists of are blocks of consecutive integers of length j placed periodically k apart. The central block sits at indices $\{-1, \ldots, -j\}$. I_0 is then added on. What adding on I_0 does is to fatten the blocks to the left of the origin by one element. Here is another way to describe the sets I_j.

Order $\mathbb{Z}$ as follows. We say $i \prec i'$ if

$$i = kt + \ell, i' = kt' + \ell'$$

and

(a) $\ell < \ell'$ or
(b) if $\ell = \ell'$, then $t < t'$.

What this does is to break $\mathbb{Z}$ into k sets according to the value of $i \bmod(k)$. If we call them $\mathbb{Z}_t = \{i \in \mathbb{Z}\colon i \bmod (k) = t\}$, then $\mathbb{Z}_t$ simply inherits the usual ordering of $\mathbb{Z}$, but all elements of $\mathbb{Z}_{t+1}$ are made greater than those of $\mathbb{Z}_t$. In these terms $I_0 = \{i\colon i \prec 0\}$ and in general

$$I_t = \{i\colon i \prec t\} - t.$$

LEMMA 7: *For $\mu \in \mathcal{M}(\Sigma, \sigma)$,*

$$(1)\ h_\mu(\sigma^k, P_1) = h_\mu(P_1 | \bigvee_{i \in I_0} \sigma^{-i}(P_1))$$

and

$$(2)\ h_\mu(\sigma, P_1) = \frac{1}{k} \sum_{t=0}^{k-1} h_\mu(P_1 | \bigvee_{j \in I_t} \sigma^{-j}(P_1)).$$

Proof: Of course (1) is standard. We begin to prove (2) with the identity

$$\begin{aligned} h_\mu(\sigma, P_1) &= \lim_{n\to\infty} \frac{1}{nk} h_\mu(\bigvee_{i=0}^{nk-1} \sigma^{-i}(P_1)) \\ &= \lim_{n\to\infty} \frac{1}{nk} \sum_{i=0}^{nk-1} h_\mu(\sigma^{-i}(P_1) | \bigvee_{\{j:\, j \prec i, 0 \le j < nk\}} \sigma^{-j}(P_1)) \\ &= \lim_{n\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} \left(\frac{1}{n} \sum_{\substack{\{i:\, i \bmod (k) = t, \\ 0 \le i < nk\}}} h_\mu(\sigma^{-i}(P_1) | \bigvee_{\{j:\, j \prec i, 0 \le j < nk\}} \sigma^{-j}(P_1)) \right) \end{aligned}$$

Let us examine

$$\frac{1}{n} \sum_{\{i:\, i \bmod (k) = t, 0 \le i < nk\}} h_\mu(\sigma^{-i}(P_1) | \bigvee_{\{j:\, j \prec i, 0 \le j < nk\}} \sigma^{-j}(P_1)).$$

For $i = t + uk$,

$$\{j: j \prec i, 0 \le j < nk\} = (I_t \cap \{-i, -i+1, \ldots, -i+nk-1\}) + i,$$

so

$$\begin{aligned} & h_\mu(\sigma^{-i}(P_1) | \bigvee_{\{j:\, j \prec i, 0 \le j < nk\}} \sigma^{-j}(P_1)) \\ &= h_\mu(\sigma^{-i}(P_1) | \bigvee_{j \in I_t \cap \{-i, \ldots, -i+nk-1\}} \sigma^{-j-i}(P_1)) \\ &= h_\mu(P_1 | \bigvee_{j \in I_t \cap \{-i, \ldots, -i+nk-1\}} \sigma^{-j}(P_1)) \end{aligned}$$

and so

$$\begin{aligned} & \frac{1}{n} \sum_{\{i:\, i \bmod (k) = t, 0 \le i < nk\}} h_\mu(\sigma^{-i}(P_1) | \bigvee_{\{j:\, j \prec i, 0 \le j < nk\}} \sigma^{-j}(P_1)) \\ &= \frac{1}{n} \sum_{u=0}^{n-1} h_\mu(P_1) | \bigvee_{j \in I_t \cap \{-t-uk, \ldots, -t+(n-u)k\}} \sigma^{-j}(P_1)). \end{aligned}$$

Notice that if we define

$$H_\mu(m) \stackrel{\text{def}}{=} h_\mu(P_1 | \bigvee_{j\in I_t\cap\{-t-mk,\ldots,-t+mk\}} \sigma^{-j}(P_1)),$$

then this quantity decreases in m, converging to $h_\mu(P_1|\bigvee_{j\in I_t}\sigma^{-j}(P_1))$.

For each choice of u,

$$H_\mu(\max(u,n-u)) \le h_\mu(P_1 | \bigvee_{j\in I_t\cap\{-t-uk,\ldots,-t+(n-u)k\}} \sigma^{-j}(P_1)) \le H_\mu(\min(u,n-u)).$$

Once n is large enough, for most $u \in \{0,\ldots,n-1\}$, the left and right sides of this inequality will both be very close to $h_\mu(P_1|\bigvee_{j\in I_t}\sigma^{-j}(P_1))$. Thus

$$\lim_{n\to\infty}\frac{1}{n}\sum_{u=0}^{n-1} h_\mu(P_1| \bigvee_{j\in I_t\cap\{-t-uk,\ldots,-t+(n-u)k\}} \sigma^{-j}(P_1)) = h_\mu(P_1|\bigvee_{j\in I_t}\sigma^{-j}(P_1))$$

completing the proof of (2).

COROLLARY 8: *In $\mathcal{M}(\Sigma,\sigma)$, if $\mu_i \xrightarrow{i} \mu$ and $h_{\mu_i}(\sigma,P_1) \xrightarrow{i} h_\mu(\sigma,P_1)$, then for all $k\in\mathbb{N}$,*

$$h_{\mu_i}(\sigma^k,P_1) \xrightarrow{i} h_\mu(\sigma^k,P_1).$$

Proof: Note that in (2) of Lemma 7, all k terms on the right hand side are upper semi-continuous in μ. If $h_{\mu_i}(\sigma,P_1) \xrightarrow{i} h_\mu(\sigma,P_1)$, then all the terms on the right side of (2) must also converge to the corresponding terms for $h_\mu(\sigma,P_1)$. In particular, term zero must converge. But this is just $h_\mu(\sigma^k,P_1)$ by (1).

Proof of Proposition 2: Just note that if $A_i \xrightarrow{i} A$ in $\mathcal{M}_T$, then $h(T_{A_i},P|_{A_i}) \xrightarrow{i} h(T_A,P|_A)$ and Corollary 8 gives us the result.

Proof of Theorem 4: A system $(X,\mathcal{F},\mu,T)$ is K iff for all finite partitions $P=\{P_1,\ldots,P_s\}$,

$$\lim_{k\to\infty} h(T^k,P) = H(P) = -\sum_{j=1}^{s}\mu(P_j)\log_2(\mu(P_j))$$

It is enough that this be true for a countable dense family of partitions in the symmetric difference metric. Even less is actually necessary. It is enough that

$$\sup_{k\in\mathbb{N}} h(T^k,P) = H(P)$$

for a countable dense family of P to conclude that T is a K-system.

Fix a system $(X, \mathcal{F}, \mu, T)$ and let

$$\mathcal{O}(\varepsilon, P) = \{A \in \mathcal{F}\colon \text{ for some } k \in \mathbb{N}, h(T_A^k, P|_A) > (1-\varepsilon)H(P|_A)\}.$$

Proposition 2 tells us $\mathcal{O}(\varepsilon, P)$ is open. It contains the K-systems so, by [O,S], it is dense. Hence for each P,

$$K(P) = \{A \in \mathcal{F}\colon \sup_{k\in\mathbb{N}} h(T_A^k, P|_A) = H(P|_A)\}$$

is residual. Intersecting over a countable dense family of partitions P, we are left with exactly the K-systems.

THEOREM 5: *We will need the following lemma, whose proof is an easy application of Lyapunov's theorem on the range of a vector-valued measure ([Ru, Theorem 5.5]). (In fact, an approximate version of the lemma has a completely elementary proof and would suffice for our purposes here.)*

LEMMA 8: *Suppose $(X, \mathcal{F}, \mu)$ is a Lebesgue probability space, $\mathcal{G} \subset \mathcal{F}$ is a non-atomic sub-σ-algebra, P is any $\mathcal{F}$-measurable finite partition of X, and $\lambda = (\lambda_1, \dots, \lambda_n)$ is any probability vector. Then there is a $\mathcal{G}$-measurable partition Q such that $Q \perp P$ and* dist $Q = \lambda$.

Proof of Theorem 5: We start with the well-known observation that a system $(Y, \mathcal{G}, \nu, S)$ is weakly mixing precisely if (and only if), for all partitions P from some countable dense family, S has a sequence $\{n_j\}$ of mixing times with respect to P, that is $\nu(S^{-n_j} E \cap F) \to \nu(E)\nu(F)\ \forall E, F \in P$. (Note that $\{n_j\}$ is allowed to depend on P.) Accordingly we define

$$\mathcal{O}(P, N, \varepsilon) = \{A \in \mathcal{G}\colon \|\text{dist }(T_A^{-N} P|E) - \text{dist } P\| < \varepsilon\ \forall E \in P\}.$$

Clearly each $\mathcal{O}(P, N, \varepsilon)$ is open, so it will suffice to show that $\bigcup_{N>0} \mathcal{O}(P, N, \varepsilon)$ is dense for each $\mathcal{F}$-measurable finite partition P and $\varepsilon > 0$. What we will in fact show is that, given $A \in \mathcal{G}$ and $\delta > 0$, there is a $\mathcal{G}$-measurable $A' \subset A$ such that $\mu(A - A') < \delta$ and $A' \in \mathcal{O}(P, N, \varepsilon)$ for some n. Without loss of generality we can assume $A = X$, since T_A is ergodic.

We regard P as a map $P\colon X \to \Gamma$, Γ the indexing set for P. If $x \in X$ and $n > 0$, dist(x, P, n) will denote the measure on Γ given by the frequencies of symbols in the P, n-name of X, that is

$$\text{dist }(x, P, n)(\gamma) = \frac{1}{n}\sum_{i=0}^{n-1} 1_{\{\gamma\}}(P(T^i x)).$$

By the ergodic theorem, given $\delta_1 > 0$ we can find n such that

$$G = \{x: ||\text{dist }(x, P, n) - \text{dist } P|| < \delta_1\}$$

has measure

$$\mu(G) > 1 - \delta_1.$$

Choose $N \gg n$, let $h = rN$ with r large, and then let $B \in \mathcal{G}$ be the base of a Rohlin tower of height h covering all but δ_1 of X. Let

$$Q = \left(\bigvee_{i=0}^{h-1} T^{-i} P\right)\Bigg|_B$$

denote the partition of B according to P, h-names and apply Lemma 8 to obtain a $\mathcal{G}$-measurable partition

$$R: B \to \{O, \dots, n-1\}$$

such that $R \perp Q$.

Now let

$$E = \bigcap_{m=0}^{n-1} \bigcap_{s=0}^{r-1} T^{sN} \bigcap_{j=0}^{m-1} T^j(R^{-1}(m))$$

and let $A' = X - E \in \mathcal{G}$. Let

$$F = T^{-N} G \cup \left(\bigcup_{s=0}^{r-2} T^{sN} \bigcup_{j=0}^{n-1} T^j B\right),$$

so $F \subset E$ and, given $\delta_2 > 0$, $\mu(F) > 1 - \delta_2$ if δ_1 is small enough and both N and R are large enough. Note that each $T^j(q)$ $(q \in Q,\ 0 \le j \le h-1)$ is either contained in or disjoint from F. If $\tilde{q} = T^j q \subset F$, then by the definition of G and construction of A' we have

$$\text{dist }(T^{-N} P | \tilde{q}) = \text{dist }(T^N x, P, n)$$

for any $x \in \tilde{q}$. Since $T^N x \in G$ we conclude

$$||\text{dist }(T^{-N} P|\tilde{q}) - \text{dist } P|| < \delta_1.$$

Fixing $p \in P$ and averaging over all $\tilde{q}$ contained in $p \cap F$ we conclude that

$$||\text{dist }(T_{A'}^{-N} P | p \cap F) - \text{dist } P|| < \delta_1$$

and hence

$$||\text{dist }(T_{A'}^{-N} P|p) - \text{dist } P|| < \varepsilon$$

if δ_1 and δ_2 are small enough. That is, $A' \in \mathcal{O}(P, N, \varepsilon)$. Since $\mu(A') > \mu(F) > 1 - \delta_2$ we can ensure that $\mu(A') > 1 - \delta$, so we are done. ∎

Mixing?

In this development we haven't discussed the property of mixing at all. It is clear that mixing cannot always be residual in $\mathcal{M}_T$ as it is not in the zero-entropy, loosely Bernoulli class. But sometimes it is, for example whenever T has positive entropy. Is it possible for some zero-entropy T that the mixing elements of $\mathcal{M}_T$ are residual? Is there any way of telling for which T they are and for which T they are not?

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